On the CJT Formalism in Multi-Field Theories

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ABSTRACT

The issues that arise when using the Cornwall-Jackiw-Tomboulis formalism in multi-field theories are investigated. Particular attention is devoted to the interplay between temperature effects, ultraviolet structure, and the interdependence of the gap equations. Results are presented explicitly in the case of the evaluation of the finite temperature effective potential of a theory with two scalar fields which has attracted interest as a toy model for symmetry nonrestoration at high temperatures. The lowest nontrivial order of approximation of the Cornwall-Jackiw-Tomboulis effective potential is shown to lead to consistent results, which are relevant for recent studies of symmetry nonrestoration by Bimonte and Lozano.

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1 INTRODUCTION

The importance of nonperturbative techniques in ordinary ("zero-temperature" or "vacuum") fied theory is widely recognized. These techniques are even more important, sometimes providing the only consistent approach to a problem, in finite temperature field theory, which is affected by infrared problems that are not easily handled within perturbative approaches.

Over the last twenty years the Cornwall-Jackiw-Tomboulis (CJT) formalism of the effective action for composite operators[1] has been frequently used as a nonperturbative technique for the study of zero-temperature problems. More recently, in Refs.[2, 3], Pi and I advocated the use of the CJT formalism also in the investigation of problems of finite temperature field theory. Tests[4, 5] of our proposal have found that it is among the best approaches[1] to the study of the type of issues naturally arising in finite temperature field theory. In some cases the results of Refs.[2, 3] have been used as standards to which other nonperturbative techniques are compared[9, 10]. These successes of the CJT formalism are however somewhat limited (especially in the finite temperature context), since they have been obtained either within rigorous analyses of single-field[2] theories[1, 2, 4, 5] or within analyses of multi-field theories in some rather drastic approximation[1, 3]. This is a noticeable limitation since the CJT formalism can be importantly affected by the presence of more than one field, requiring the study of interdependent ultraviolet-divergent self-consistent equations.

In order to test the reliability of the approximations based on the CJT formalism in the context of multi-field theories, in the present paper I derive the "bubble approximation" of the CJT effective potential in a (thermal) two-scalar-field theory, with particular attention to the interplay between temperature effects, ultraviolet structure, and the interdependence of the gap equations.

Establishing more rigorously the reliability of the CJT formalism in multi-field (thermal) theories can be very important; for example, this could be useful for the investigation of the possibility of symmetry nonrestoration at high temperatures[11-13] a proposal of great importance for modern cosmology and particle physics, which has been recently reenergized by the results presented in Refs.[14-18]. Symmetry nonrestoration scenarios usually require a multi-field theory, and their investigation has lead to some controversy (recently revisited in Ref.[16]); in fact, the results on this subject obtained within certain nonperturbative approximation schemes[19] are very different from the ones obtained perturbatively[11-18]. The CJT formalism appears to be ideally suited for the investigation of this subject, since it encodes some features of the nonperturbative regime within a systematic expansion in loops which is quite similar to those encountered in perturbative approaches; it might therefore provide the possibility to bridge the conceptual and quantitative differences between perturbative and nonperturbative approaches to the study of symmetry nonrestoration.

In the CJT formalism the (thermal) effective potential V_T is obtained as the solution of a variational problem for the effective potential for composite operators W_T:

$$V_T(\phi) = W_T[\phi; D_T(\phi; k)] \ , \tag{1}$$

$$\left[\frac{\delta W_T[\phi; G(k)]}{\delta G(k)}\right]_{G(k)=D_T(\phi;k)} = 0 \ , \tag{2}$$

where the index T stands for temperature ($T=0$ corresponds to the theory in vacuum). A rigorous definition of W_T can be found in Refs.[1, 2, 3, 20]; for the purposes of the present

[1] In particular, the approach discussed in Refs.[6, 7, 8] was found to have comparable qualities.

[2] Notice that the study of the $O(N)$ model in the large N approximation, as done for example in Ref.[1], effectively reduces the analysis to the one of a single-field theory.

paper it is sufficient to observe that W_T admits a loop expansion, with $G(k)$ appearing as the (dressed) propagator:

$$W_T[\phi; G(k)] = V_{tree}(\phi) + \frac{1}{2}\sum\!\!\!\!\!\!\int_k^{(T)} Tr[\ln G^{-1}(k) + D^{-1}_{tree}(\phi;k)G(k) - 1] + W^*_T[\phi; G(k)] \ , \quad (3)$$

where W^*_T is given by all the two-particle-irreducible vacuum-to-vacuum graphs with two or more loops in the theory with vertices given by the interaction part of the shifted ($\Phi \to \Phi+\phi$) Lagrangian and propagators set equal to $G(k)$. Also notice that, when $T \neq 0$, the fourth component of momentum is discretized, $k_4 = i\pi nT$ (n is even for bosons, whereas it is odd for fermions), as appropriate for the imaginary time formalism of finite temperature field theory, which I intend to use. Moreover, in order to be able to discuss at once the zero-temperature and the finite-temperature cases, I introduced the notation

$$\sum\!\!\!\!\!\!\int_p^{(T)} \equiv T \sum_{n=-\infty}^{\infty} \int \frac{d^3p}{(2\pi)^3} \ , \quad (4)$$

which at $T=0$ is understood to denote the usual momentum integration of field theory in vacuum

$$\sum\!\!\!\!\!\!\int_p^{(0)} \equiv \int \frac{d^4p}{(2\pi)^4} \ . \quad (5)$$

The bubble approximation, which I consider in this paper, is the lowest nontrivial order[1-3,20,21] of approximation of the effective potential in the CJT formalism. It is obtained by including in W^*_T only the "double-bubble diagrams", *i.e.* diagrams with the topology of two rings touching at one point. Formally, the bubble effective potential can be written as

$$\begin{aligned} V^{bubble}_T(\phi) &= W^{bubble}_T[\phi; D^{bubble}_T(\phi;k)] = V_{tree}(\phi) + \frac{1}{2}\sum\!\!\!\!\!\!\int_k^{(T)} \ln[D^{bubble}_T(k)]^{-1} \\ &+ \frac{1}{2}\sum\!\!\!\!\!\!\int_k^{(T)} [D^{-1}_{tree}(\phi;k)D^{bubble}_T(k) - 1] + W^{*bubble}_T[\phi; D^{bubble}_T(\phi;k)] \ , \quad (6) \end{aligned}$$

where D^{bubble}_T is the solution of

$$\left[\frac{\delta W^{bubble}_T[\phi; G(k)]}{\delta G(k)}\right]_{G(k)=D^{bubble}_T(\phi;k)} = 0 \ . \quad (7)$$

A complete discussion of the bubble approximation and some of its applications of physical relevance are given in Refs.[1-3,20,21]. For the present paper it is important that, for single-field theories, it has been possible to show the renormalizability and general consistency of the bubble approximation, and it is therefore reasonable to test the reliability of the CJT formalism for multi-field theories by performing an analogous calculation. As a preliminary indication of the importance of the CJT formalism for the study of symmetry nonrestoration, I also point out that the CJT bubble approximation leads to the equations on which the analysis of Ref.[15] is based.

In order to introduce modularly the various conceptual and technical issues, the paper is organized as follows. In the next section, I review the classic zero-temperature result for the single-scalar-field Z_2-invariant model, with quartic contact interactions. In Sec.3, I consider a two-scalar-field $Z_2 \times Z_2$-invariant model at zero temperature, and address the issues introduced by the interdependence among the corresponding two gap equations (one for each field) of the CJT formalism. In Sec.4, I finally add thermal effects, and investigate the resulting structure of the bubble effective potential in the same two-field model considered in Sec.3. Sec.5 is devoted to closing remarks.

2 Z_2 MODEL AT $T = 0$

The single scalar field theory of Euclidean Lagrange density

$$L = \frac{1}{2}(\partial_\mu \Phi)(\partial^\mu \Phi) + \frac{1}{2} m^2 \Phi^2 + \frac{\lambda_\Phi}{24}\Phi^4 \ , \tag{8}$$

has been extensively studied within the bubble approximation of the CJT formalism. I review its analysis in order to set up notation and make observations useful in the study of the more complex model considered in the later sections.

The interaction Lagrangian of the $\Phi \to \Phi + \phi$ shifted theory is

$$L_{int}(\phi; \Phi) = \frac{\lambda_\Phi}{24}\Phi^4 + \frac{\lambda_\Phi}{6}\phi\Phi^3 \ , \tag{9}$$

the tree-level (classical) potential has the form

$$V_{tree} = \frac{m^2}{2}\phi^2 + \frac{\lambda_\Phi}{24}\phi^4 \ , \tag{10}$$

which reflects the Z_2 invariance ($\Phi \to -\Phi$) of (8), and the tree-level propagator is

$$D_{tree}(\phi; k) = \frac{1}{k^2 + m^2 + \frac{\lambda_\Phi}{2}\phi^2} \ . \tag{11}$$

From Eqs.(6)-(9) one finds that the zero-temperature CJT bubble[3] potential is given by[2]

$$\begin{aligned} V_0(\phi) \ = \ & \frac{m^2}{2}\phi^2 + \frac{\lambda_\Phi}{24}\phi^4 + \frac{1}{2}\sum\!\!\!\!\!\!\!\int_k^{(0)} \ln D_0^{-1}(\phi; k) \\ & + \frac{1}{2}\sum\!\!\!\!\!\!\!\int_k^{(0)} [(k^2 + m^2 + \frac{\lambda_\Phi}{2}\phi^2) D_0(\phi; k) - 1] + \frac{\lambda_\Phi}{8}\left[\sum\!\!\!\!\!\!\!\int_k^{(0)} D_0(\phi; k)\right]^2 \ . \end{aligned} \tag{12}$$

where $D_0(\phi; k)$ is the solution of the bubble-approximated gap equation (7), which in the present case can be written as

$$D_0^{-1}(\phi; k) = k^2 + m^2 + \frac{\lambda_\Phi}{2}\phi^2 + \frac{\lambda_\Phi}{2}\sum\!\!\!\!\!\!\!\int_p^{(0)} D_0(\phi; p) \ . \tag{13}$$

The last term on the r.h.s. of Eq.(12) (which is responsible for the last term on the r.h.s. of Eq.(13)) is the contribution of the double-bubble diagram[2], which is the leading two-loop contribution to the CJT effective potential for composite operators in this model.

Without loss of generality one can write

$$D_0(\phi; k) = \frac{1}{k^2 + M_0^2(\phi; k)} \ , \tag{14}$$

[3]Notice that in the following, unlike in the introduction, I suppress the "bubble" index.

and in terms of the "effective mass" M_0 the gap equation Eq.(13) can be written as

$$M_0^2(\phi;k) = m^2 + \frac{\lambda_\Phi}{2}\phi^2 + \frac{\lambda_\Phi}{2} P_0[M_0] \ , \tag{15}$$

where $P_0[M_0]$ is the zero-temperature limit of

$$P_T[M_T] \equiv \sum\!\!\!\!\!\!\!\int_p^{(T)} \frac{1}{p^2 + M_T^2(\phi;p)} \ . \tag{16}$$

Since $P_0[M_0]$ is momentum independent, Eq.(15) implies that (in the bubble approximation) the effective mass is momentum independent: $M_0 = M_0(\phi)$.

In terms of the solution $M_0(\phi)$ of Eq.(15), the bubble effective potential takes the form

$$\begin{aligned} V_0(\phi) \ = \ & \frac{m^2}{2}\phi^2 + \frac{\lambda_\Phi}{24}\phi^4 + \frac{1}{2}\sum\!\!\!\!\!\!\!\int_k^{(0)} \ln[k^2 + M_0^2(\phi)] \\ & -\frac{1}{2}\,[M_0^2(\phi) - m^2 - \frac{\lambda_\Phi}{2}\phi^2]\, P_0[M_0(\phi)] + \frac{1}{8}\lambda_\Phi \left(P_0[M_0(\phi)]\right)^2 \end{aligned} \tag{17}$$

This expression of V_0 is affected by two types of divergencies: one originating from its divergent integrals, and the other originating from the fact that $M_0(\phi)$ is not well-defined because of the infinities in $P_0(M_0)$. Let me start the renormalization procedure by obtaining a well-defined finite expression for $M_0(\phi)$. As shown in Ref.[22],

$$P_0[M_0] = I_1 - M_0^2 I_2 + \frac{M_0^2}{16\pi^2} \ln\frac{M_0^2}{\mu^2} \ , \tag{18}$$

where $I_{1,2}$ are divergent integrals

$$I_1 \equiv \int \frac{d^3k}{(2\pi)^3}\,\frac{1}{2|\mathbf{k}|} = \lim_{\Lambda\to\infty} \frac{\Lambda^2}{8\pi^2} \ , \tag{19}$$

$$I_2 \equiv \int \frac{d^3k}{(2\pi)^3}\,[\frac{1}{2|\mathbf{k}|} - \frac{1}{2\sqrt{|\mathbf{k}|^2 + \mu^2}}] = \lim_{\Lambda\to\infty} \frac{1}{16\pi^2} \ln\frac{\Lambda^2}{\mu^2} \ , \tag{20}$$

μ is the renormalization scale, and Λ is the ultraviolet momentum cut-off.

Using Eq.(18), the gap equation can be rewritten as

$$M_0^2 = I_1 - M_0^2 I_2 + m^2 + \frac{\lambda_\Phi}{2}\phi^2 + \lambda_\Phi \frac{M_0^2}{32\pi^2} \ln\frac{M_0^2}{\mu^2} \ , \tag{21}$$

and the divergent terms can be reabsorbed by introducing the following renormalized parameters $\widetilde{\lambda}_\Phi$ and $\widetilde{m}$

$$\begin{aligned} \frac{1}{\widetilde{\lambda}_\Phi} \ &= \ \frac{1}{\lambda_\Phi} + \frac{I_2}{2} \ , & (22) \\ \frac{\widetilde{m}}{\widetilde{\lambda}_\Phi} \ &= \ \frac{m^2}{\lambda_\Phi} + \frac{I_1}{2} \ , & (23) \end{aligned}$$

leading to the renormalized gap equation

$$M_0^2 = \widetilde{m}^2 + \frac{\widetilde{\lambda}_\Phi}{2}\phi^2 + \widetilde{\lambda}_\Phi \frac{M_0^2}{32\pi^2} \ln \frac{M_0^2}{\mu^2} \; . \tag{24}$$

Before completing the renormalization of V_0, let me discuss the structure of the renormalized parameters that were just introduced. In particular, notice that, in order to keep the renormalized coupling $\widetilde{\lambda}_\Phi$ positive and finite, the bare coupling must take negatively vanishing values as the cut-off is removed ($\lambda_\Phi \to 0^-$ as $\Lambda \to \infty$), leading to an unstable[23] theory. This is one aspect of the known "triviality" of the theory under consideration; in fact, Eq.(22) also implies that the theory becomes free ($\widetilde{\lambda}_\Phi \to 0$) as the cut-off is removed, if, as required by stability, the bare coupling is positive. For physical applications, in which it is desirable to keep positive both λ_Φ and $\widetilde{\lambda}_\Phi$, this Z_2 model is usually considered as an effective low-energy theory, with finite cut-off Λ such that

$$\frac{\widetilde{\lambda}_\Phi}{32\pi^2} \ln \frac{\Lambda^2}{\mu^2} < 1 \; , \tag{25}$$

(as required by Eq.(22) for positive λ_Φ and $\widetilde{\lambda}_\Phi$), but larger than any physical mass scale in the problem (momenta, temperature, etc.). Actually, in many applications[2] the interesting case is

$$\frac{\widetilde{\lambda}_\Phi}{32\pi^2} \ln \frac{\Lambda^2}{\mu^2} << 1 \; , \tag{26}$$

which leads to the ideal scenario of cut-off independence[4] with positive λ_Φ and $\widetilde{\lambda}_\Phi$. Consistently with these observations I am ultimately most interested in the cases (25)-(26), and I keep track of the ultraviolet cut-off Λ. Renormalizability is obviously encoded in the finiteness of the $\Lambda \to \infty$ limit.

Having clarified these "triviality-related issues", I can proceed verifying that the relations (22)-(23), which were introduced to renormalize the bubble gap equation, also renormalize the bubble effective potential. In the simple model presently under consideration this can be done in several ways[2]; I adopt one that can be rather naturally generalized, as shown in the following sections, to the case of multi-field theories. Let me start by noticing that from the known[22] result

$$\sum\!\!\!\!\!\!\!\int_k^{(0)} \ln[k^2 + M_0^2] = \frac{M_0^2}{2} I_1 - \frac{M_0^4}{4} I_2 + \frac{M_0^4}{64\pi^2}[\ln \frac{M_0^2}{\mu^2} - \frac{1}{2}] \; , \tag{27}$$

and Eqs.(15) and (17), it follows that (up to irrelevant ϕ-independent contributions)

$$\begin{aligned} V_0 \;\; = \;\; & \frac{m^2}{2}\phi^2 + \frac{\lambda_\Phi}{24}\phi^4 + \frac{M_0^4}{64\pi^2}[\ln \frac{M_0^2}{\mu^2} - \frac{1}{2}] - \frac{M_0^4}{4} I_2 + \frac{M_0^2}{2} I_1 \\ & - \frac{1}{2\lambda_\Phi}[M_0^2 - m^2 - \frac{\lambda_\Phi}{2}\phi^2]^2 \; . \end{aligned} \tag{28}$$

This can be rewritten using the definitions (22)-(23) as

$$\begin{aligned} V_0 \;\; = \;\; & \frac{\widetilde{m}^2}{2}\phi^2 + \frac{\widetilde{\lambda}_\Phi - \lambda_\Phi}{12}\phi^4 + \frac{M_0^4}{64\pi^2}[\ln \frac{M_0^2}{\mu^2} - \frac{1}{2}] \\ & - \frac{1}{2\widetilde{\lambda}_\Phi}[M_0^2 - \widetilde{m}^2 - \frac{\widetilde{\lambda}_\Phi}{2}\phi^2]^2 \; . \end{aligned} \tag{29}$$

[4]As shown in Ref.[2], and reviewed below, the cut-off decouples from the analysis in the limit (26).

Finally, using the renormalized gap equation, one finds that

$$
\begin{aligned}
V_0 &= \frac{\widetilde{m}^2}{2}\phi^2 + \frac{\widetilde{\lambda}_\Phi}{24}\phi^4 + \frac{\widetilde{\lambda}_\Phi - \lambda_\Phi}{12}\phi^4 \\
&\quad + \frac{M_0^4}{64\pi^2}[\ln \frac{M_0^2}{\mu^2} - \frac{1}{2}] - \frac{\widetilde{\lambda}_\Phi}{2}\,[\frac{M_0^2}{32\pi^2}\ln \frac{M_0^2}{\mu^2}]^2 \; ,
\end{aligned}
\tag{30}
$$

where M_0 is the solution of the renormalized gap equation (24).

The dependence on the cut-off is all included in the term

$$
\frac{\widetilde{\lambda}_\Phi - \lambda_\Phi}{12}\phi^4 = - \frac{\frac{\widetilde{\lambda}_\Phi}{32\pi^2}\ln \frac{\Lambda^2}{\mu^2}}{1 - \frac{\widetilde{\lambda}_\Phi}{32\pi^2}\ln \frac{\Lambda^2}{\mu^2}} \frac{\widetilde{\lambda}_\Phi}{12}\phi^4 \; . \tag{31}
$$

The renormalizability of the CJT bubble effective potential of the Z_2 model is therefore shown by the fact that the $\Lambda \to \infty$ limit of (31) is well-defined and finite. The form of the effective potential in the limit (26) is obtained from (30) by neglecting the term (31).

3 $Z_2 \times Z_2$ MODEL AT $T = 0$

Still keeping, for the moment, $T=0$, I now study the two-scalar-field theory of Euclidean Lagrange density

$$
L = \frac{1}{2}(\partial_\mu \Phi)(\partial^\mu \Phi) + \frac{1}{2}(\partial_\mu \Psi)(\partial^\mu \Psi) + \frac{1}{2}m^2\Phi^2 + \frac{1}{2}\omega^2\Psi^2 + \frac{\lambda_\Phi}{24}\Phi^4 + \frac{\lambda_\Psi}{24}\Psi^4 + \frac{\lambda_{\Phi\Psi}}{4}\Phi^2\Psi^2 \; , \tag{32}
$$

which is $Z_2 \times Z_2$ invariant $[(\Phi \to -\Phi) \times (\Psi \to -\Psi)]$.

In general in such a theory one could consider the effective potential $V(\phi, \psi)$ corresponding to the shifts $\{\Phi, \Psi\} \to \{\Phi + \phi, \Psi + \psi\}$. However, for the type of test of the CJT formalism that I am performing it is sufficient to look at the projection of $V(\phi, \psi)$ on the $\psi = 0$ (or equivalently the $\phi = 0$) axis, and this is convenient in order to simplify the rather bulky formulas involved. Moreover, scenarios for symmetry nonrestoration at high temperatures within this $Z_2 \times Z_2$ model require $\lambda_\Psi > -\lambda_{\Phi\Psi} > \lambda_\Phi > 0$ (or, alternatively, $\lambda_\Phi > -\lambda_{\Phi\Psi} > \lambda_\Psi > 0$), in which case all the significant information is encoded in $V(\phi, \psi = 0)$ (or, alternatively, $V(\phi = 0, \psi)$). Therefore, in the following, I concentrate on $V(\phi, \psi = 0)$, *i.e.* shifts $\{\Phi, \Psi\} \to \{\Phi + \phi, \Psi\}$, and, for short, use the notation $V(\phi)$ for (the bubble approximation of) $V(\phi, \psi = 0)$.

The shift $\{\Phi, \Psi\} \to \{\Phi + \phi, \Psi\}$ leads to the interaction Lagrangian

$$
L_{int}(\phi; \Phi) = \frac{\lambda_\Phi}{24}\Phi^4 + \frac{\lambda_\Psi}{24}\Psi^4 + \frac{\lambda_\Phi}{6}\phi\Phi^3 + \frac{\lambda_{\Phi\Psi}}{2}\phi\Phi^3 \; , \tag{33}
$$

the tree-level potential

$$
V_{tree} = \frac{m^2}{2}\phi^2 + \frac{\lambda_\Phi}{24}\phi^4 \; , \tag{34}
$$

and the tree-level propagator

$$
[D_{tree}(\phi; k)]_{ab} = \frac{\delta_{a1}\delta_{b1}}{k^2 + m^2 + \frac{\lambda_\Phi}{2}\phi^2} + \frac{\delta_{a2}\delta_{b2}}{k^2 + \omega^2 + \frac{\lambda_{\Phi\Psi}}{2}\phi^2} \; . \tag{35}
$$

The zero-temperature bubble effective potential is given by

$$
\begin{aligned}
V_0 \;=\; & \frac{m^2}{2}\phi^2 + \frac{\lambda_\Phi}{24}\phi^4 - \frac{1}{2}\sum\!\!\!\!\!\!\!\int_k^{(0)} \{\ln[D_0(\phi;k)]_{11} + \ln[D_0(\phi;k)]_{22}\} \\
& + \frac{1}{2}\sum\!\!\!\!\!\!\!\int_k^{(0)} \{(k^2{+}m^2{+}\frac{\lambda_\Phi}{2}\phi^2)[D_0(\phi;k)]_{11} + (k^2{+}\omega^2{+}\frac{\lambda_{\Phi\Psi}}{2}\phi^2)[D_0(\phi;k)]_{22} - 2\} \\
& + \frac{\lambda_\Phi}{8}\left[\sum\!\!\!\!\!\!\!\int_k^{(0)} [D_0(\phi;k)]_{11}\right]^2 + \frac{\lambda_\Psi}{8}\left[\sum\!\!\!\!\!\!\!\int_k^{(0)} [D_0(\phi;k)]_{22}\right]^2 \\
& + \frac{\lambda_{\Phi\Psi}}{4}\sum\!\!\!\!\!\!\!\int_k^{(0)} [D_0(\phi;k)]_{11}\sum\!\!\!\!\!\!\!\int_p^{(0)} [D_0(\phi;p)]_{22} \;. \qquad (36)
\end{aligned}
$$

where $[D_0(\phi;k)]_{11}$ and $[D_0(\phi;k)]_{22}$ are the solutions of the gap equations

$$
\begin{aligned}
([D_0(\phi;k)]_{11})^{-1} \;&=\; ([D_{tree}(\phi;k)]_{11})^{-1} + \frac{\lambda_\Phi}{2}\sum\!\!\!\!\!\!\!\int_p^{(0)} [D_0(p)]_{11} + \frac{\lambda_{\Psi\Psi}}{2}\sum\!\!\!\!\!\!\!\int_p^{(0)} [D_0(p)]_{22} \;, \\
([D_0(\phi;k)]_{22})^{-1} \;&=\; ([D_{tree}(\phi;k)]_{22})^{-1} + \frac{\lambda_\Psi}{2}\sum\!\!\!\!\!\!\!\int_p^{(0)} [D_0(p)]_{22} + \frac{\lambda_{\Psi\Psi}}{2}\sum\!\!\!\!\!\!\!\int_p^{(0)} [D_0(p)]_{11} \;. \qquad (37)
\end{aligned}
$$

Again, it is convenient to reexpress the effective propagator D_0 in terms of effective masses

$$
[D_0(\phi;k)]_{ab} = \frac{\delta_{a1}\delta_{b1}}{k^2 + M_0^2(\phi;k)} + \frac{\delta_{a2}\delta_{b2}}{k^2 + \Omega_0^2(\phi;k)} \;, \qquad (38)
$$

allowing to rewrite the gap equations as

$$
\begin{aligned}
M_0^2(\phi;k) \;&=\; m^2 + \frac{\lambda_\Phi}{2}\phi^2 + \frac{\lambda_\Phi}{2}P_0[M_0] + \frac{\lambda_{\Phi\Psi}}{2}P_0[\Omega_0] \;, \\
\Omega_0^2(\phi;k) \;&=\; \omega^2 + \frac{\lambda_{\Phi\Psi}}{2}\phi^2 + \frac{\lambda_\Psi}{2}P_0[\Omega_0] + \frac{\lambda_{\Phi\Psi}}{2}P_0[M_0] \;. \qquad (39)
\end{aligned}
$$

This shows that also in this two-field theory the effective masses are momentum independent within the bubble approximation: $M_0 = M_0(\phi)$, $\Omega_0 = \Omega_0(\phi)$.

In terms of effective masses and bare parameters, V_0 has the form

$$
\begin{aligned}
V_0 \;=\; & \frac{m^2}{2}\phi^2 + \frac{\lambda_\Phi}{24}\phi^4 + \frac{1}{2}\sum\!\!\!\!\!\!\!\int_k^{(0)} \{\ln[k^2 + M_0^2(\phi)] + \ln[k^2 + \Omega_0^2(\phi)]\} \\
& - \frac{1}{2}\,[M_0^2(\phi){-}m^2{-}\frac{\lambda_\Phi}{2}\phi^2]\,P_0[M_0] - \frac{1}{2}\,[\Omega_0^2(\phi){-}\omega^2{-}\frac{\lambda_{\Phi\Psi}}{2}\phi^2]\,P_0[\Omega_0] \\
& + \frac{\lambda_\Phi}{8}\,(P_0[M_0])^2 + \frac{\lambda_\Psi}{8}\,(P_0[\Omega_0])^2 + \frac{\lambda_{\Phi\Psi}}{4}P_0[M_0]P_0[\Omega_0] \;. \qquad (40)
\end{aligned}
$$

The first step toward the renormalization of V_0 is the renormalization of the gap equations, which, using Eq.(18), can be rewritten as

$$
\begin{aligned}
M_0^2(\phi;k) \;=\; & m^2 + \frac{\lambda}{2}\phi^2 + \frac{\lambda_\Phi}{2}\left(I_1 - M_0^2 I_2 + \frac{M_0^2}{16\pi^2}\ln\frac{M_0^2}{\mu^2}\right) \\
& + \frac{\lambda_{\Phi\Psi}}{2}\left(I_1 - \Omega_0^2 I_2 + \frac{\Omega_0^2}{16\pi^2}\ln\frac{\Omega_0^2}{\mu^2}\right) \;,
\end{aligned}
$$

$$
\begin{aligned}
\Omega_0^2(\phi;k) &= \omega^2 + \frac{\lambda_{\Phi\Psi}}{2}\phi^2 + \frac{\lambda_\Psi}{2}\left(I_1 - \Omega_0^2 I_2 + \frac{\Omega_0^2}{16\pi^2}\ln\frac{\Omega_0^2}{\mu^2}\right) \\
&\quad + \frac{\lambda_{\Phi\Psi}}{2}\left(I_1 - M_0^2 I_2 + \frac{M_0^2}{16\pi^2}\ln\frac{M_0^2}{\mu^2}\right) \ . \qquad (41)
\end{aligned}
$$

Notice that the interdependence of the gap equations affects importantly the structure of divergent terms. Since in each gap equation divergent coefficients appear in front of both M_0^2 and Ω_0^2, in this two-field theory the renormalization cannot proceed by considering the gap equations independently (whereas the only gap equation present in the single-field theory considered earlier could obviously be renormalized on its own). Nevertheless, I am able to obtain renormalized results by exploiting the fact that combining appropriately the gap Eqs.(41) one can derive the following equivalent set of equations

$$
\begin{aligned}
0 &= \frac{\phi^2}{2} + \frac{M_0^2}{32\pi^2}\ln\frac{M_0^2}{\mu^2} + \frac{I_1}{2} + \frac{\lambda_\Psi m^2 - \lambda_{\Phi\Psi}\omega^2}{\lambda_\Phi\lambda_\Psi - \lambda_{\Phi\Psi}^2} \\
&\quad - \left(\frac{I_2}{2} + \frac{\lambda_\Psi}{\lambda_\Phi\lambda_\Psi - \lambda_{\Phi\Psi}^2}\right) M_0^2 + \frac{\lambda_{\Phi\Psi}}{\lambda_\Phi\lambda_\Psi - \lambda_{\Phi\Psi}^2}\Omega_0^2 \ , \\
0 &= \frac{\Omega_0^2}{32\pi^2}\ln\frac{\Omega_0^2}{\mu^2} + \frac{I_1}{2} + \frac{\lambda_\Phi\omega^2 - \lambda_{\Phi\Psi}m^2}{\lambda_\Phi\lambda_\Psi - \lambda_{\Phi\Psi}^2} \\
&\quad - \left(\frac{I_2}{2} + \frac{\lambda_\Phi}{\lambda_\Phi\lambda_\Psi - \lambda_{\Phi\Psi}^2}\right)\Omega_0^2 + \frac{\lambda_{\Phi\Psi}}{\lambda_\Phi\lambda_\Psi - \lambda_{\Phi\Psi}^2} M_0^2 \ . \qquad (42)
\end{aligned}
$$

Notice that in the first (second) of these equations divergent coefficients appear only in front of M_0 (Ω_0). The structure of the Eqs.(42) suggests the introduction of renormalized parameters $\widetilde{\lambda}_\Phi$, $\widetilde{\lambda}_\Psi$, $\widetilde{\lambda}_{\Phi\Psi}$, $\widetilde{m}$, $\widetilde{\omega}$, defined by

$$
\frac{\widetilde{\lambda}_\Psi}{\widetilde{\lambda}_\Phi\widetilde{\lambda}_\Psi - \widetilde{\lambda}_{\Phi\Psi}^2} = \frac{I_2}{2} + \frac{\lambda_\Psi}{\lambda_\Phi\lambda_\Psi - \lambda_{\Phi\Psi}^2} \ , \qquad (43)
$$

$$
\frac{\widetilde{\lambda}_\Phi}{\widetilde{\lambda}_\Phi\widetilde{\lambda}_\Psi - \widetilde{\lambda}_{\Phi\Psi}^2} = \frac{I_2}{2} + \frac{\lambda_\Phi}{\lambda_\Phi\lambda_\Psi - \lambda_{\Phi\Psi}^2} \ , \qquad (44)
$$

$$
\frac{\widetilde{\lambda}_{\Phi\Psi}}{\widetilde{\lambda}_\Phi\widetilde{\lambda}_\Psi - \widetilde{\lambda}_{\Phi\Psi}^2} = \frac{\lambda_{\Phi\Psi}}{\lambda_\Phi\lambda_\Psi - \lambda_{\Phi\Psi}^2} \ , \qquad (45)
$$

$$
\frac{\widetilde{\lambda}_\Phi\widetilde{\omega}^2 - \widetilde{\lambda}_{\Phi\Psi}\widetilde{m}^2}{\widetilde{\lambda}_\Phi\widetilde{\lambda}_\Psi - \widetilde{\lambda}_{\Phi\Psi}^2} = \frac{I_1}{2} + \frac{\lambda_\Psi m^2 - \lambda_{\Phi\Psi}\omega^2}{\lambda_\Phi\lambda_\Psi - \lambda_{\Phi\Psi}^2} \ , \qquad (46)
$$

$$
\frac{\widetilde{\lambda}_\Psi\widetilde{m}^2 - \widetilde{\lambda}_{\Phi\Psi}\widetilde{\omega}^2}{\widetilde{\lambda}_\Phi\widetilde{\lambda}\Psi - \widetilde{\lambda}_{\Phi\Psi}^2} = \frac{I_1}{2} + \frac{\lambda_\Psi m^2 - \lambda_{\Phi\Psi}\omega^2}{\lambda_\Phi\lambda_\Psi - \lambda_{\Phi\Psi}^2} \ , \qquad (47)
$$

In terms of these renormalized parameters the Eqs.(42) can be rewritten as

$$
\begin{aligned}
0 &= \frac{\phi^2}{2} + \frac{M_0^2}{32\pi^2}\ln\frac{M_0^2}{\mu^2} + \frac{\widetilde{\lambda}_\Psi\widetilde{m}^2 - \widetilde{\lambda}_{\Phi\Psi}\widetilde{\omega}^2 - \widetilde{\lambda}_\Psi M_0^2 + \widetilde{\lambda}_{\Phi\Psi}\Omega_0^2}{\widetilde{\lambda}_\Phi\widetilde{\lambda}_\Psi - \widetilde{\lambda}_{\Phi\Psi}^2} \ , \\
0 &= \frac{\Omega_0^2}{32\pi^2}\ln\frac{\Omega_0^2}{\mu^2} + \frac{\widetilde{\lambda}_\Phi\widetilde{\omega}^2 - \widetilde{\lambda}_{\Phi\Psi}\widetilde{m}^2 - \widetilde{\lambda}_\Phi\Omega_0^2 + \widetilde{\lambda}_{\Phi\Psi}M_0^2}{\widetilde{\lambda}_\Phi\widetilde{\lambda}\Psi - \widetilde{\lambda}_{\Phi\Psi}^2} \ . \qquad (48)
\end{aligned}
$$

In turn these equations can be combined to obtain the following equivalent set of equations

$$
\begin{aligned}
M_0^2(\phi;k) &= \widetilde{m}^2 + \frac{\widetilde{\lambda}_\Phi}{2}\phi^2 + \frac{\widetilde{\lambda}_\Phi}{2}\frac{M_0^2}{16\pi^2}\ln\frac{M_0^2}{\mu^2} + \frac{\widetilde{\lambda}_{\Phi\Psi}}{2}\frac{\Omega_0^2}{16\pi^2}\ln\frac{\Omega_0^2}{\mu^2} \ , \\
\Omega_0^2(\phi;k) &= \widetilde{\omega}^2 + \frac{\widetilde{\lambda}_{\Phi\Psi}}{2}\phi^2 + \frac{\widetilde{\lambda}_\Psi}{2}\frac{\Omega_0^2}{16\pi^2}\ln\frac{\Omega_0^2}{\mu^2} + \frac{\widetilde{\lambda}_{\Phi\Psi}}{2}\frac{M_0^2}{16\pi^2}\ln\frac{M_0^2}{\mu^2} \ ,
\end{aligned}
\tag{49}
$$

which can be interpreted as renormalized gap equations.

Before completing the renormalization of V_0, I want to observe that also the $Z_2 \times Z_2$ model presently under consideration presents triviality-related features. In fact, from (43)-(47) it follows that by requiring

$$
\widetilde{\lambda}_\Psi > 0 \ , \quad \widetilde{\lambda}_\Phi > 0 \ , \quad \widetilde{\lambda}_\Psi\widetilde{\lambda}_\Phi - \widetilde{\lambda}_{\Phi\Psi}^2 > 0 \tag{50}
$$

(the conditions usually assumed[11, 15] to be sufficient to ensure the stability of the theory), one finds that $\lambda_\Psi \to 0^-$ and $\lambda_\Phi \to 0^-$ in the $\Lambda \to \infty$ limit, whereas by insisting on a bare potential bounded from below, *i.e.* demanding

$$
\lambda_\Psi > 0 \ , \quad \lambda_\Phi > 0 \ , \quad \lambda_\Psi\lambda_\Phi - \lambda_{\Phi\Psi}^2 > 0 \ , \tag{51}
$$

one finds that $\widetilde{\lambda}_\Psi \to 0^+$ and $\widetilde{\lambda}_\Phi \to 0^+$ in the $\Lambda \to \infty$ limit.

A physically meaningful $Z_2 \times Z_2$ model can be certainly obtained as a low-energy effective theory with cut-off Λ small enough to be consistent with both (50) and (51), so, like in the previous section, I keep track of the ultraviolet cut-off Λ, and take the limit $\Lambda \to \infty$ only when testing renormalizability.

In order to show that the renormalization prescriptions (43)-(47) also renormalize the effective potential, I rewrite V_0 using Eqs.(27), (40), and (42)

$$
\begin{aligned}
V_0 = &\frac{m^2}{2}\phi^2 + \frac{\lambda}{24}\phi^4 - \frac{M_0^4+\Omega_0^4}{4}I_2 + \frac{M_0^2+\Omega_0^2}{2}I_1 + \frac{M_0^4}{64\pi^2}[\ln\frac{M_0^2}{\mu^2} - \frac{1}{2}] + \frac{\Omega_0^4}{64\pi^2}[\ln\frac{\Omega_0^2}{\mu^2} - \frac{1}{2}] \\
&-\frac{\lambda_\Phi}{2}\left[\frac{\lambda_{\Phi\Psi}(\Omega_0^2-\omega^2) - \lambda_\Psi(M_0^2-m^2) + (\lambda_\Phi\lambda_\Psi - \lambda_{\Phi\Psi})\frac{\phi^2}{2}}{\lambda_\Phi\lambda_\Psi - \lambda_{\Phi\Psi}^2}\right]^2 \\
&-\frac{\lambda_\Psi}{2}\left[\frac{\lambda_{\Phi\Psi}(M_0^2-m^2) - \lambda_\Phi(\Omega_0^2-\omega^2)}{\lambda_\Phi\lambda_\Psi - \lambda_{\Phi\Psi}^2}\right]^2 \\
&-\lambda_{\Phi\Psi}\frac{[\lambda_{\Phi\Psi}(\Omega_0^2-\omega^2) - \lambda_\Psi(M_0^2-m^2) + (\lambda_\Phi\lambda_\Psi - \lambda_{\Phi\Psi})\frac{\phi^2}{2}][\lambda_{\Phi\Psi}(M_0^2-m^2) - \lambda_\Phi(\Omega_0^2-\omega^2)]}{[\lambda_\Phi\lambda_\Psi - \lambda_{\Phi\Psi}^2]^2} \ .
\end{aligned}
\tag{52}
$$

Using the definitions (43)-(47) one then finds that

$$
\begin{aligned}
V_0 = &\frac{\widetilde{m}^2}{2}\phi^2 + \frac{\widetilde{\lambda}_\Phi}{24}\phi^4 + \frac{\widetilde{\lambda}_\Phi - \lambda_\Phi}{12}\phi^4 + \frac{M_0^4}{64\pi^2}[\ln\frac{M_0^2}{\mu^2} - \frac{1}{2}] + \frac{\Omega_0^4}{64\pi^2}[\ln\frac{\Omega_0^2}{\mu^2} - \frac{1}{2}] \\
&-\frac{\widetilde{\lambda}_\Phi}{2}\left[\frac{\widetilde{\lambda}_{\Phi\Psi}(\Omega_0^2-\omega^2) - \widetilde{\lambda}_\Psi(M_0^2-m^2) + (\widetilde{\lambda}_\Phi\widetilde{\lambda}_\Psi - \widetilde{\lambda}_{\Phi\Psi})\frac{\phi^2}{2}}{\widetilde{\lambda}_\Phi\widetilde{\lambda}_\Psi - \widetilde{\lambda}_{\Phi\Psi}^2}\right]^2 \\
&-\frac{\widetilde{\lambda}_\Psi}{2}\left[\frac{\widetilde{\lambda}_{\Phi\Psi}(M_0^2-m^2) - \widetilde{\lambda}_\Phi(\Omega_0^2-\omega^2)}{\widetilde{\lambda}_\Phi\widetilde{\lambda}_\Psi - \widetilde{\lambda}_{\Phi\Psi}^2}\right]^2 \\
&-\widetilde{\lambda}_{\Phi\Psi}\frac{[\widetilde{\lambda}_{\Phi\Psi}(\Omega_0^2-\omega^2) - \widetilde{\lambda}_\Psi(M_0^2-m^2) + (\widetilde{\lambda}_\Phi\widetilde{\lambda}_\Psi - \widetilde{\lambda}_{\Phi\Psi})\frac{\phi^2}{2}][\widetilde{\lambda}_{\Phi\Psi}(M_0^2-m^2) - \widetilde{\lambda}_\Phi(\Omega_0^2-\omega^2)]}{\widetilde{\lambda}_\Phi\widetilde{\lambda}_\Psi - \widetilde{\lambda}_{\Phi\Psi}^2} \ ,
\end{aligned}
\tag{53}
$$

which using Eqs.(48) finally leads to the result

$$
\begin{aligned}
V_0 \;=\; & \frac{\widetilde{m}^2}{2}\phi^2 + \frac{\widetilde{\lambda}_\Phi}{24}\phi^4 + \frac{\widetilde{\lambda}_\Phi - \lambda_\Phi}{12}\phi^4 + \frac{M_0^4}{64\pi^2}[\ln\frac{M_0^2}{\mu^2} - \frac{1}{2}] + \frac{\Omega_0^4}{64\pi^2}[\ln\frac{\Omega_0^2}{\mu^2} - \frac{1}{2}] \\
& -\frac{\widetilde{\lambda}_\Phi}{8}[\frac{M_0^2}{16\pi^2}\ln\frac{M_0^2}{\mu^2}]^2 - \frac{\widetilde{\lambda}_\Psi}{8}[\frac{\Omega_0^2}{16\pi^2}\ln\frac{\Omega_0^2}{\mu^2}]^2 - \frac{\widetilde{\lambda}_{\Phi\Psi}}{4}[\frac{M_0^2}{16\pi^2}\ln\frac{M_0^2}{\mu^2}][\frac{\Omega_0^2}{16\pi^2}\ln\frac{\Omega_0^2}{\mu^2}] \; . \qquad (54)
\end{aligned}
$$

Again, the only Λ-dependent contribution comes from the term $(\widetilde{\lambda}_\Phi - \lambda_\Phi)\phi^4/12$. As it can be easily derived from Eqs.(43)-(47), this term has a well-defined and finite $\Lambda \to \infty$ limit, indicating that the CJT bubble potential of the $Z_2 \times Z_2$ model at zero temperature is renormalizable.

4 $Z_2 \times Z_2$ MODEL AT FINITE T

In this section I finally consider the finite temperature case; specifically, I study the CJT bubble effective potential of the $Z_2 \times Z_2$ model in the imaginary time formalism of finite temperature field theory. The temperature dependence, besides introducing additional elements of technical difficulty, affects very importantly some of the issues under investigation in the present paper. In ordinary analyses of perturbative renomarlizability of the finite temperature effective potential a central role is played by the fact the renormalization prescriptions for the parameters of a field theory are temperature independent[20, 24]. However, as shown below, because of the temperature dependence of the effective propagator (which results from the nonperturbative nature of the approach) there are finite temperature Feynman diagrams relevant for the CJT formalism that give highly nontrivial temperature- and ϕ-dependent divergent contributions to the effective potential. Unless these divergencies can be reabsorbed by the introduction of temperature-independent renormalized parameters, the physical consistency of the nonperturbative approach is to be doubted[20].

I start by introducing temperature-dependent effective masses

$$
[D_T(\phi;k)]_{ab} = \frac{\delta_{a1}\delta_{b1}}{k^2 + M_T^2(\phi;k)} + \frac{\delta_{a2}\delta_{b2}}{k^2 + \Omega_T^2(\phi;k)} \; , \qquad (55)
$$

which, in the bubble approximation, must satisfy the gap equations

$$
\begin{aligned}
M_T^2(\phi;k) &= m^2 + \frac{\lambda_\Phi}{2}\phi^2 + \frac{\lambda_\Phi}{2}P_T[M_T] + \frac{\lambda_{\Phi\Psi}}{2}P_T[\Omega_T] \; , \\
\Omega_T^2(\phi;k) &= \omega^2 + \frac{\lambda_{\Phi\Psi}}{2}\phi^2 + \frac{\lambda_\Psi}{2}P_T[\Omega_T] + \frac{\lambda_{\Phi\Psi}}{2}P_T[M_T] \; , \qquad (56)
\end{aligned}
$$

indicating that M_T and Ω_T are momentum independent: $M_T = M_T(\phi)$, $\Omega_T = \Omega_T(\phi)$.

In terms of the effective masses the bubble effective potential can be written as

$$
\begin{aligned}
V_T \;=\; & \frac{m^2}{2}\phi^2 + \frac{\lambda_\Phi}{24}\phi^4 + \frac{1}{2}\sum\!\!\!\!\!\!\int_k^{(T)} \{\ln[k^2 + M_T^2] + \ln[k^2 + \Omega_T^2]\} \\
& -\frac{1}{2}\,[M_T^2 - m^2 - \frac{\lambda_\Phi}{2}\phi^2]\,P_T[M_T] - \frac{1}{2}\,[\Omega_T^2 - \omega^2 - \frac{\lambda_{\Phi\Psi}}{2}\phi^2]\,P_T[\Omega_T] \\
& +\frac{\lambda_\Phi}{8}\,(P_T[M_T])^2 + \frac{\lambda_\Psi}{8}\,(P_T[\Omega_T])^2 + \frac{\lambda_{\Phi\Psi}}{4}P_T[M_T]P_T[\Omega_T]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{m^2}{2}\phi^2 + \frac{\lambda_\Phi}{24}\phi^4 + \frac{1}{2}\sum\!\!\!\!\!\!\int_k^{(T)} \{\ln[k^2+M_T^2] + \ln[k^2+\Omega_T^2]\} \\
&\quad -\frac{\lambda_\Phi}{8}\left(P_T[M_T]\right)^2 - \frac{\lambda_\Psi}{8}\left(P_T[\Omega_T]\right)^2 - \frac{\lambda_{\Phi\Psi}}{4} P_T[M_T] P_T[\Omega_T] \;, \qquad (57)
\end{aligned}
$$

where the last equality follows from the gap equations (56).

The ultraviolet divergent contributions can be identified using the well-known results[25] for the "tadpole"

$$
\begin{aligned}
P_T[M] &= I_1 - M^2 I_2 + P_T^{(f)}[M] \\
P_T^{(f)}[M] &\equiv \frac{M^2}{16\pi^2}\ln\frac{M^2}{\mu^2} - \int \frac{d^3k}{(2\pi)^3}\left[\sqrt{|\mathbf{k}|^2+M^2}\left(1-exp\left(\frac{\sqrt{|\mathbf{k}|^2+M^2}}{T}\right)\right)\right]^{-1} \;, \qquad (58)
\end{aligned}
$$

and the "one loop"

$$
\begin{aligned}
\sum\!\!\!\!\!\!\int_k^{(T)} \ln[k^2+M^2] &= -\frac{M^4}{4}I_2 + \frac{M^2}{2}I_1 + Q_T^{(f)}[M] \\
Q_T^{(f)}[M] &\equiv \frac{M^4}{64\pi^2}[\ln\frac{M^2}{\mu^2} - \frac{1}{2}] + T\int \frac{d^3k}{(2\pi)^3}\, \ln\left[1-exp\left(\frac{\sqrt{|\mathbf{k}|^2+M^2}}{T}\right)\right] \;. \qquad (59)
\end{aligned}
$$

$P_T^{(f)}$ and $Q_T^{(f)}$ are finite (*i.e.* do not diverge as the cut-off is removed) and their structure is known[25] within a "high temperature" (small M/T) expansion

$$
\begin{aligned}
P_T^{(f)}[M] &\simeq \frac{T^2}{12} - \frac{MT}{4\pi} + \frac{M^2}{16\pi^2}\ln\frac{M^2}{T^2} \\
Q_T^{(f)}[M] &\simeq -\frac{\pi^2 T^4}{90} + \frac{M^2T^2}{24} - \frac{M^3T}{12\pi} - \frac{M^4}{64\pi^2}\ln\frac{M^2}{T^2} \;. \qquad (60)
\end{aligned}
$$

Eqs.(57)-(60) show that some of the divergent contributions to V_T come from terms that are analogous to those encountered in the zero-temperature case, but now involve the temperature dependent effective masses M_T,Ω_T in place of their zero-temperature limits M_0,Ω_0. Other divergent contributions to V_T come from products of two P_T's, and involve terms of structure $P_T^{(f)}I_1$ or $P_T^{(f)}I_2$. The analysis of this latter type of divergent contributions is complicated by the fact that the information available on $P_T^{(f)}$ is only in the form of a series expansion. However, I show below that the steps for renormalization that I followed in the previous sections can be generalized to the finite temperature case in such a way to require no explicit information on $P_T^{(f)}$ (besides the fact that it stays finite as the cut-off is removed).

First, let me reexpress the gap equations (56) in the spirit of the Eqs.(42):

$$
\begin{aligned}
0 &= \frac{\phi^2}{2} + P_T^{(f)}(M_T) + \frac{I_1}{2} + \frac{\lambda_\Psi m^2 - \lambda_{\Phi\Psi}\omega^2}{\lambda_\Phi\lambda_\Psi - \lambda_{\Phi\Psi}^2} \\
&\quad -\left(\frac{I_2}{2} - \frac{\lambda_\Psi}{\lambda_\Phi\lambda_\Psi - \lambda_{\Phi\Psi}^2}\right) M_T^2 + \frac{\lambda_{\Phi\Psi}}{\lambda_\Phi\lambda_\Psi - \lambda_{\Phi\Psi}^2}\Omega_T^2 \;, \\
0 &= P_T^{(f)}(\Omega_T) + \frac{I_1}{2} + \frac{\lambda_\Phi\omega^2 - \lambda_{\Phi\Psi}m^2}{\lambda_\Phi\lambda_\Psi - \lambda_{\Phi\Psi}^2} \\
&\quad -\left(\frac{I_2}{2} - \frac{\lambda_\Phi}{\lambda_\Phi\lambda_\Psi - \lambda_{\Phi\Psi}^2}\right)\Omega_T^2 + \frac{\lambda_{\Phi\Psi}}{\lambda_\Phi\lambda_\Psi - \lambda_{\Phi\Psi}^2} M_T^2 \;. \qquad (61)
\end{aligned}
$$

In agreement with the physical arguments[20, 24] indicating that the renormalization prescriptions for the parameters of a thermal field theory should be temperature-independent, the divergencies in these equations can be absorbed by introducing the same renormalized parameters found necessary in the zero-temperature analysis; in fact, in terms of the $\widetilde{m}$, $\widetilde{\omega}$, $\widetilde{\lambda}_\Phi$, $\widetilde{\lambda}_\Psi$, $\widetilde{\lambda}_{\Phi\Psi}$ of Eqs.(43)-(47), the Eqs.(61) can be rewritten as

$$
\begin{aligned}
0 &= \frac{\phi^2}{2} + P_T^{(f)}(M_T) + \frac{\widetilde{\lambda}_\Psi \widetilde{m}^2 - \widetilde{\lambda}_{\Phi\Psi}\widetilde{\omega}^2 - \widetilde{\lambda}_\Psi M_T^2 + \widetilde{\lambda}_{\Phi\Psi}\Omega_T^2}{\widetilde{\lambda}_\Phi \widetilde{\lambda}_\Psi - \widetilde{\lambda}_{\Phi\Psi}^2} ~, \\
0 &= P_T^{(f)}(\Omega_T) + \frac{\widetilde{\lambda}_\Phi \widetilde{\omega}^2 - \widetilde{\lambda}_{\Phi\Psi}\widetilde{m}^2 + \widetilde{\lambda}_{\Phi\Psi} M_T^2 - \widetilde{\lambda}_\Phi \Omega_T^2}{\widetilde{\lambda}_\Phi \widetilde{\lambda}_\Psi - \widetilde{\lambda}_{\Phi\Psi}^2} ~.
\end{aligned} \tag{62}
$$

These can then be recombined to obtain the "renormalized gap equations"

$$
\begin{aligned}
M_T^2 &= \widetilde{m}^2 + \frac{\widetilde{\lambda}_\Phi}{2}\phi^2 + \frac{\widetilde{\lambda}_\Phi}{2} P_T^{(f)}(M_T) + \frac{\widetilde{\lambda}_{\Phi\Psi}}{2} P_T^{(f)}(\Omega_T) ~, \\
\Omega_T^2 &= \widetilde{\omega}^2 + \frac{\widetilde{\lambda}_{\Phi\Psi}}{2}\phi^2 + \frac{\widetilde{\lambda}_\Psi}{2} P_T^{(f)}(\Omega_T) + \frac{\widetilde{\lambda}_{\Phi\Psi}}{2} P_T^{(f)}(M_T) ~.
\end{aligned} \tag{63}
$$

In order to show that temperature independent renormalization prescriptions also allow to renormalize the bubble effective potential, let me start by observing that Eqs.(57)-(59) imply

$$
\begin{aligned}
V_T &= \frac{m^2}{2}\phi^2 + \frac{\lambda_\Phi}{24}\phi^4 + Q_T^{(f)}[M_T] + Q_T^{(f)}[\Omega_T] - \frac{M_T^4 + \Omega_T^4}{4} I_2 - \frac{M_T^2 + \Omega_T^2}{2} I_1 \\
&\quad - \frac{\lambda_\Phi}{8}\left(I_1 - M_T^2 I_2 + P_T^{(f)}[M_T]\right)^2 - \frac{\lambda_\Psi}{8}\left(I_1 - \Omega_T^2 I_2 + P_T^{(f)}[\Omega_T]\right)^2 \\
&\quad - \frac{\lambda_{\Phi\Psi}}{4}\left(I_1 - M_T^2 I_2 + P_T^{(f)}[M_T]\right)\left(I_1 - \Omega_T^2 I_2 + P_T^{(f)}[\Omega_T]\right) ~.
\end{aligned} \tag{64}
$$

One can then use Eq.(61) to show that

$$
\begin{aligned}
V_T = {} & \frac{m^2}{2}\phi^2 + \frac{\lambda}{24}\phi^4 - \frac{M_T^4 + \Omega_T^4}{4} I_2 + \frac{M_T^2 + \Omega_T^2}{2} I_1 + Q_T^{(f)}[M_T] + Q_T^{(f)}[\Omega_T] \\
& - \frac{\lambda_\Phi}{2}\left[\frac{\lambda_{\Phi\Psi}(\Omega_T^2 - \omega^2) - \lambda_\Psi(M_T^2 - m^2) + (\lambda_\Phi\lambda_\Psi - \lambda_{\Phi\Psi})\frac{\phi^2}{2}}{\lambda_\Phi\lambda_\Psi - \lambda_{\Phi\Psi}^2}\right]^2 \\
& - \frac{\lambda_\Psi}{2}\left[\frac{\lambda_{\Phi\Psi}(M_T^2 - m^2) - \lambda_\Phi(\Omega_T^2 - \omega^2)}{\lambda_\Phi\lambda_\Psi - \lambda_{\Phi\Psi}^2}\right]^2 \\
& - \lambda_{\Phi\Psi}\frac{[\lambda_{\Phi\Psi}(\Omega_T^2 - \omega^2) - \lambda_\Psi(M_T^2 - m^2) + (\lambda_\Phi\lambda_\Psi - \lambda_{\Phi\Psi})\frac{\phi^2}{2}][\lambda_{\Phi\Psi}(M_T^2 - m^2) - \lambda_\Phi(\Omega_T^2 - \omega^2)]}{[\lambda_\Phi\lambda_\Psi - \lambda_{\Phi\Psi}^2]^2} ~,
\end{aligned} \tag{65}
$$

which in terms of the renormalized parameters of Eqs.(43)-(47) takes the form

$$
\begin{aligned}
V_T = {} & \frac{\widetilde{m}^2}{2}\phi^2 + \frac{\widetilde{\lambda}_\Phi}{24}\phi^4 + \frac{\widetilde{\lambda}_\Phi - \lambda_\Phi}{12}\phi^4 + Q_T^{(f)}[M_T] + Q_T^{(f)}[\Omega_T] \\
& - \frac{\widetilde{\lambda}_\Phi}{2}\left[\frac{\widetilde{\lambda}_{\Phi\Psi}(\Omega_T^2 - \omega^2) - \widetilde{\lambda}_\Psi(M_T^2 - m^2) + (\widetilde{\lambda}_\Phi\widetilde{\lambda}_\Psi - \widetilde{\lambda}_{\Phi\Psi})\frac{\phi^2}{2}}{\widetilde{\lambda}_\Phi\widetilde{\lambda}_\Psi - \widetilde{\lambda}_{\Phi\Psi}^2}\right]^2
\end{aligned}
$$

$$
-\frac{\widetilde{\lambda}_\Psi}{2}\left[\frac{\widetilde{\lambda}_{\Phi\Psi}(M_T^2-m^2)-\widetilde{\lambda}_\Phi(\Omega_T^2-\omega^2)}{\widetilde{\lambda}_\Phi\widetilde{\lambda}_\Psi-\widetilde{\lambda}_{\Phi\Psi}^2}\right]^2
$$

$$
-\widetilde{\lambda}_{\Phi\Psi}\frac{[\widetilde{\lambda}_{\Phi\Psi}(\Omega_T^2-\omega^2)-\widetilde{\lambda}_\Psi(M_T^2-m^2)+(\widetilde{\lambda}_\Phi\widetilde{\lambda}_\Psi-\widetilde{\lambda}_{\Phi\Psi})\frac{\phi^2}{2}][\widetilde{\lambda}_{\Phi\Psi}(M_T^2-m^2)-\widetilde{\lambda}_\Phi(\Omega_T^2-\omega^2)]}{\widetilde{\lambda}_\Phi\widetilde{\lambda}_\Psi-\widetilde{\lambda}_{\Phi\Psi}^2}\ , \quad (66)
$$

and can be finally reexpressed using the renormalized gap equations (63) as

$$
\begin{aligned}
V_T \ = \ & \frac{\widetilde{m}^2}{2}\phi^2+\frac{\widetilde{\lambda}_\Phi}{24}\phi^4+\frac{\widetilde{\lambda}_\Phi-\lambda_\Phi}{12}\phi^4+Q_T^{(f)}[M_T]+Q_T^{(f)}[\Omega_T] \\
& -\frac{\widetilde{\lambda}_\Phi}{8}(P_T^{(f)}[M_T])^2-\frac{\widetilde{\lambda}_\Psi}{8}(P_T^{(f)}[\Omega_T])^2-\frac{\widetilde{\lambda}_{\Phi\Psi}}{4}P_T^{(f)}[M_T]P_T^{(f)}[\Omega_T]\ ,
\end{aligned} \quad (67)
$$

Once again the Λ-dependence is confined to the term $(\widetilde{\lambda}_\Phi-\lambda_\Phi)\phi^4/12$ which [as easily checked from Eqs.(43)-(47)] has a well-defined and finite $\Lambda\to\infty$ limit, indicating the renormalizability of the CJT bubble potential of the $Z_2\times Z_2$ model at finite temperature.

5 CLOSING REMARKS

In this paper I have provided evidence in support of the reliability of approximations based on the CJT formalism in the context of multi-field (thermal) theories. The aspects of the analysis which required the development of new technical tools, such as the simultaneous renormalization of the gap equations and the triviality-related issues, might encode more physical information than I was able to uncover here; this should be investigated in the future. It would also be interesting to check whether in multi-field theories it is possible to confirm the agreement established within single-field theories between the nonperturbative approximations used in Refs.[9, 10] and the CJT bubble approximation. Issues such as those related to the interdependence of the gap equations are not easily phrased in some of the other nonperturbative approximations, and this might lead to complications.

Most importantly, the apparent reliability of the CJT formalism for multi-field theories can be exploited in the investigation of physical problems, especially the possibility of symmetry nonrestoration, for which, as discussed in the introduction, the CJT formalism is ideally suited. The analysis presented in this paper already provides results directly relevant for symmetry nonrestoration; in particular, from Eqs.(60) and (63) it follows that at high temperatures

$$
\begin{aligned}
M_T^2 \ &\simeq \ \widetilde{m}^2+\frac{\widetilde{\lambda}_\Phi}{2}\phi^2+\frac{\widetilde{\lambda}_\Phi}{2}(\frac{T^2}{12}-\frac{M_T T}{4\pi})+\frac{\widetilde{\lambda}_{\Phi\Psi}}{2}(\frac{T^2}{12}-\frac{\Omega_T T}{4\pi})\ , \\
\Omega_T^2 \ &\simeq \ \widetilde{\omega}^2+\frac{\widetilde{\lambda}_{\Phi\Psi}}{2}\phi^2+\frac{\widetilde{\lambda}_\Psi}{2}(\frac{T^2}{12}-\frac{\Omega_T T}{4\pi})+\frac{\widetilde{\lambda}_{\Phi\Psi}}{2}(\frac{T^2}{12}-\frac{M_T T}{4\pi})\ ,
\end{aligned} \quad (68)
$$

which are the equations used in the symmetry nonrestoration analysis of Ref.[15]. In Ref.[15] these equations were taken as a starting point, without a discussion of the issues related to ultraviolet divergences; therefore, my analysis, by providing an explicit renormalization procedure leading to (68), renders more robust those results. On the other hand my investigation of the ultraviolet structure raises some new issues for symmetry nonrestoration. Firstly, it appears that the physical consistency of the analysis requires that the theory be considered as an effective low-energy theory. Moreover, the triviality-related issues encountered

in Sec.3, indicate that it is necessary to reconsider the conventional assumption that (50) be sufficient for stability. It appears, in fact, necessary to perform a more careful stability analysis analogous to the one given in Ref.[23], in which it was shown that, as a by-product of triviality, in the simple Z_2 model (the $\lambda\phi^4$ model), the condition $\widetilde{\lambda}_\Phi > 0$ [whose equivalent in the $Z_2 \times Z_2$ model is Eq.(50)] is insufficient for stability. Further investigation of this issue, and its relevance for the possibility of symmetry nonrestoration, is left for future work.

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References

[1] J.M. Cornwall, R. Jackiw, and E. Tomboulis, Phys. Rev. D10, 2428 (1974).

[2] G. Amelino-Camelia and S.-Y. Pi, Phys. Rev. D47 (1993) 2356.

[3] G. Amelino-Camelia, Phys. Rev. D49 (1994) 2740.

[4] C.G. Boyd, D.E. Brahm and S.D.H. Hsu, Phys. Rev. D48 (1993) 4963.

[5] M. Quiros, hep-ph/9304284, 4th Hellenic School on Elementary Particle Physics, Corfu, Greece, 2-20 Sep 1992.

[6] D. Kirzhnits and A. Linde, Ann. Phys. 101 (1976) 195.

[7] M. Dine, R.G. Leigh, P. Huet, A. Linde, and D. Linde, Phys. Rev. D46, 550 (1992); Phys. Lett. B283 (1992) 319.

[8] C.G. Boyd, D.E. Brahm and S.D.H. Hsu, Phys. Rev. D48 (1993) 4952.

[9] H. Lee, K. Na, and J. H. Yee, Phys. Rev. D51 (1995) 3125.

[10] A. Mishra and H. Mishra, *Effective potential at finite temperature: A variational approach*, Ahmedabad Physical Research Laboratory preprint; H. Nakkagawa and H. Yokota, *RG Improvement of the Effective Potential at Finite Temperature*, Nara University preprint.

[11] S. Weinberg, Phys. Rev. **D9** (1974) 3357.

[12] R.N. Mohapatra and G. Senjanovic, Phys. Rev. Lett. 42 (1979) 1651; Phys. Rev. D20 (1979) 3390.

[13] P. Langacker and S.-Y. Pi, Phys. Rev. Lett. 45 (1980) 1.

[14] G. Dvali, A. Melfo, and G. Senjanovic, Phys. Rev. Lett. 75 (1995) 4559.

[15] G. Bimonte and G. Lozano, Phys. Lett. B366 (1996) 248.

[16] G. Bimonte and G. Lozano, Nucl. Phys. B460 (1996) 155.

[17] T. G. Roos, hep-th/9511073, *Wilson Renormalization Group Study of Inverse Symmetry Breaking* Cornell preprint.

[18] G. Dvali, A. Melfo, and G. Senjanovic, hep-ph/9601376, *Nonrestoration of Spontaneously Broken P and CP at High Temperature*, ICTP preprint; G. Dvali and K.Tamvakis, hep-ph/9602336, *Symmetry Nonrestoration at High Temperature and Supersymmetry*, ICTP preprint.

[19] Y. Fujimoto and S. Sakakibara, Phys. Lett. B151 (1985) 260; E. Manesis and S. Sakakibara, Phys. Lett. B157 (1985) 287; G.A. Hajj and P.N. Stevenson, Phys. Rev. D413 (1988) 413; K.G.Klimenko, Z.Phys. C43 (1989) 581; Theor. Math. Phys. 80 (1989) 929.

[20] R. Jackiw and G. Amelino-Camelia, hep-ph/9311324, in *Proceedings of the Third Workshop on Thermal Field Theories and Their Applications, Banff, Canada, August 15-27, 1993*, edited by F.C. Khanna, R. Kobes, G. Kunstatter, and H. Umezawa (World Scientific, 1994).

[21] P. Castorina, M. Consoli, and D. Zappala, Phys. Lett. B201 (1988) 90.

[22] R. Jackiw, Phys. Rev. D 9 (1974) 1686.

[23] W.A. Bardeen and Moshe Moshe, Phys. Rev. D28 (1983) 1372.

[24] J.I. Kapusta, *Finite-Temperature Field Theory*, (Cambridge University Press, 1989).

[25] L. Dolan and R. Jackiw, Phys. Rev. D 9 (1974) 3320.